

Schwinger-Dyson Equation in Minkowski Space beyond the IE Approximation

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ABSTRACT

We investigate properties of mass functions for fermion in Quantum Electrodynamics calculated by Schwinger-Dyson equation in strong coupling region.

Numerical results for the mass functions in which one-loop integration is performed in Minkowski space. Calculated results without the instantaneous exchange approximation are compared with those obtained in Euclidean space.

§1. Introduction

The chiral phase transition has been studied by various methods. One such method is implementation of Schwinger-Dyson equation (SDE),¹⁾ which can evaluate non-perturbative phenomena. So far, many works for chiral symmetry breaking have been done with the SDE in momentum representation, in which one-loop contribution is integrated over Euclidean space.

Some calculations of the mass function for fermion with the SDE have been done in Minkowski space. In Ref.2), spectral representation for Green-functions are assumed, in which the mass functions are calculated in Lorentz invariant form.

In order to extend the SDE at finite temperature, we need to integrate the energy and the momentum separately due to existence of the Boltzmann factor. In Ref.3), the mass function is analyzed in Euclidean space at zero temperature in which the energy and the momentum are integrated separately.

At finite temperature for equilibrium systems, the imaginary-time formalism (ITF) is implemented, which continue to Euclidean space at zero temperature limit.

On the other hand, the real-time formalism (RTF) for non-equilibrium systems is formulated in Minkowski space. The SDE in RTF has been studied with instantaneous exchange approximation (IEA),⁴⁾ in which gauge boson energy is neglected. In IEA, the mass function does not dependent on the energy and the critical coupling of chiral symmetry breaking is about half of one calculated in Euclidean space at zero temperature.⁵⁾

Analytic continuation from Euclidean space to Minkowski space is valid in perturbative calculation if pole positions in complex plane of energy are known. However it is not trivial in non-perturbative region.

In this paper, we study mass functions of fermion with SDE in Minkowski space beyond the IEA in Abelian gauge theory, such as Quantum Electrodynamics (QED). Before studying at finite temperature, we first study zero temperature limit and we examine deviation from results obtained by IEA.

In section 2, we formulate the SDE in Minkowski space. In section 3, some numerical results are shown and calculated results are compared with those obtained by SDE in Euclidean space. Section 4 is devoted to summary and some comments.

§2. SDE for mass function of fermion

We calculate a fermion self-energy $\Sigma(P)$ in QED in 4-dimensions, which is given by

$$-i\Sigma(P) = (-ie)^2 \int \frac{d^4Q}{(2\pi)^4} \gamma^\mu iS(Q) \Gamma^\nu iD_{\mu\nu}(K), \quad (2.1)$$

where $S(Q)$ and $D_{\mu\nu}(K)$ are propagators of a quark with momentum $Q = (q_0, \mathbf{q})$ and a photon with momentum $K = P - Q = (k_0, \mathbf{k})$, respectively. Here, $P = (p_0, \mathbf{p})$ is a external momentum of the fermion.

The fermion propagator is given by

$$iS(Q) = \frac{iZ}{\not{Q} - \Sigma(Q) + i\varepsilon} = \frac{i}{A(Q)\not{Q} - B(Q) + i\varepsilon} \quad (2.2)$$

In this paper, we calculate a real part of mass function $M = \text{tr}(\text{Re}\Sigma)/4$ with $\text{Im}(\Sigma) = 0$ in the Landau gauge, in which the wavefunction renormalization constant is $Z = 1$ in one-loop order of perturbation. Therefore, we calculate the self-energy Eq.(2.1) with $A = 1$, $\Sigma = B$ and the fermion-photon vertex with $\Gamma_\mu = \gamma_\mu$. Here, the photon propagator is given as

$$iD_{\mu\nu}(K) = \left(-g_{\mu\nu} + \frac{K_\mu K_\nu}{K^2} \right) \frac{i}{K^2 + i\epsilon} \quad (2.3)$$

in the Landau gauge. The mass function is written by

$$M(P) \equiv M_1(P) + M_2(P) \quad (2.4)$$

with

$$M_1(P) = -e^2 \int \frac{d^4 Q}{(2\pi)^4} 3M(Q) \mathcal{P} \left[\frac{1}{k^2} \right] \pi \delta(Q^2 - M^2(Q)) \quad (2.5)$$

and

$$M_2(P) = -e^2 \int \frac{d^4 Q}{(2\pi)^4} 3M(Q) \mathcal{P} \left[\frac{1}{Q^2 - M^2(Q)} \right] \pi \delta(K^2). \quad (2.6)$$

Here, $\mathcal{P}[\]$ denotes a principal part of the propagator. In the IEA, $k_0 = 0$ is assumed, in which the virtuality of photon is $K^2 = -\mathbf{k}^2 < 0$. In order to examine deviation from the IEA, we consider cases for $K^2 < 0$ with $k_0^2 \neq 0$, which gives $M_2 = 0$ due to the delta-function $\delta(K^2)$ in Eq.(2.6).

Integrating over the energy q_0 and the angle between \mathbf{p} and \mathbf{q} in Eq.(2.5), we obtain

$$M_1(p_0, p) = \frac{3\alpha}{4\pi} \sum_i \int_{\Lambda_0}^{\Lambda} dq \frac{q}{p} M(q_0, q) \log \left| \frac{k_0^2 - \zeta_+}{k_0^2 - \zeta_-} \right| \frac{1}{|2q_0 - \frac{\partial}{\partial q_0} M^2|} \Big|_{q_0=q_0^{(i)}} \quad (2.7)$$

with $\alpha = e^2/(4\pi)$ and $\zeta_{\pm} = (p \pm q)^2$, where we define $p = |\mathbf{p}|$, $q = |\mathbf{q}|$, and $k_0 = p_0 - q_0$. Here, Λ and Λ_0 denote cut-off parameters of q integration.

In the IEA, the mass function M_1 does not depend on energy p_0 . Therefore, we substitute the solution $q_0 = \pm \sqrt{q^2 + M_1^2} = \pm E$ into the integrand of Eq.(2.7). The mass function in IEA is given by

$$M_1^{(IEA)}(p_0, p) = \frac{3\alpha}{2\pi} \int_{\Lambda_0}^{\Lambda} dq \frac{q}{p} M_1(q_0, q) \log \left(\frac{\zeta_+}{\zeta_-} \right) \frac{1}{2E} \quad (2.8)$$

However, beyond the IEA, the mass function M depends on the energy p_0 . Therefore, an equation $f(q_0) = q_0^2 - q^2 - M^2(q_0, q) = 0$ should be solved in M_1 for each q .

We solve the SDE by iteration method. In our calculation, we solve the equation $f(q_0) = 0$ numerically and substitute the solutions $q_0 = q_0^{(i)}$ into the integrand of Eq.(2.7) for each iteration step.

In Euclidean space, the SDE for the mass function is given by

$$M_E^{(3)}(p_4, p) = \frac{3\alpha}{4\pi^2} \int dq_4 dq \frac{q}{p} \log \left(\frac{k_4^2 + \zeta_+}{k_4^2 + \zeta_-} \right) \frac{M_E^{(3)}(q_4, q)}{Q_E^2 + (M_E^{(3)}(q_4, q))^2} \quad (2.9)$$

with $k_4 = p_4 - q_4$ and $Q_E^2 = q_4^2 + q^2$, which has been studied in Ref.3).

§3. Numerical results

In this section, some numerical results are presented. The momentum q is integrated over $0.01\Lambda \leq q \leq \Lambda$.

In order to study deviation from the IEA, we introduce a parameter a with $0 \leq a \leq 1$ and we replace k_0^2 by ak_0^2 in Eq.(2.7). Here, $a = 0$ corresponds to the IEA.

Furthermore, we consider a case with $k_0^2 \leq \zeta_-$ in Minkowski space, which corresponds to a contribution of the virtual photon with space-like virtuality. In this region, the integrand of the mass function M_1 is positive definite and $M_2 = 0$.

In Fig.1, the convergence of M_1/Λ integrated over $-4\Lambda \leq p_0 \leq 4\Lambda$ and $0.01\Lambda \leq p \leq \Lambda$ are presented for $a = 0.0, 0.5, 1.0$ at $\alpha = 2$. Here, plus symbols represent mass functions calculated by Eq.(2.10) in Euclidean space.

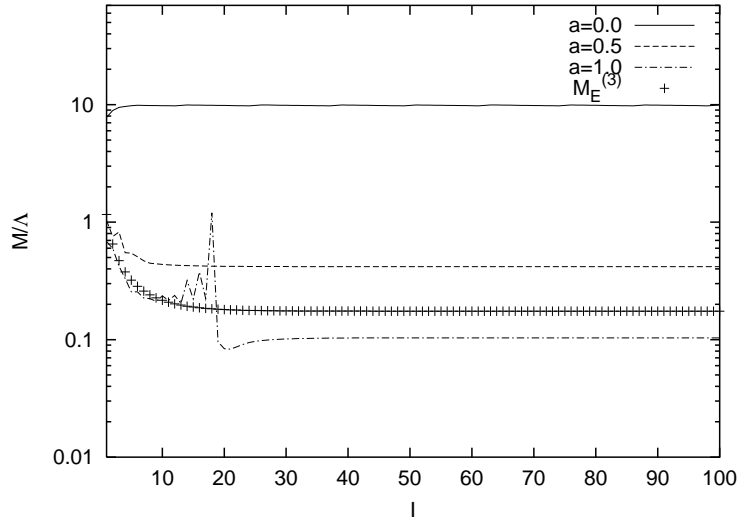


Fig. 1. The convergence of M/Λ with $\alpha = 2$ integrated over $-4\Lambda \leq p_0 \leq 4\Lambda$ and $0.01\Lambda \leq p \leq \Lambda$ with $a = 0.0, 0.5, 1.0$, respectively. The horizontal axis denotes number of iterations. The plus symbols represent calculated results for $M_E^{(3)}$.

In Fig.2, the p_0 dependences of M_1/Λ integrated over $0.01\Lambda \leq p \leq \Lambda$ with $a = 0.0, 0.05, 0.5, 1.0$ are presented. In this figure, the plus symbols represent the result for $M_E^{(3)}$.

In Fig.3, critical couplings of chiral symmetry breaking for various cases ^{*)} are presented, in which $M^{(4)}$ denotes a mass function calculated by⁶⁾

$$M_E^{(4)}(P_E^2) = \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dQ_E^2 \frac{M_E^{(4)}(Q_E^2)}{Q_E^2 + \left(M_E^{(4)}(Q_E^2)\right)^2} \frac{2Q_E^2}{P_E^2 + Q_E^2 + \sqrt{(P_E^2 - Q_E^2)^2}}, \quad (3.1)$$

where Q_E^2 and P_E^2 are squared four-momenta in Euclidean space.

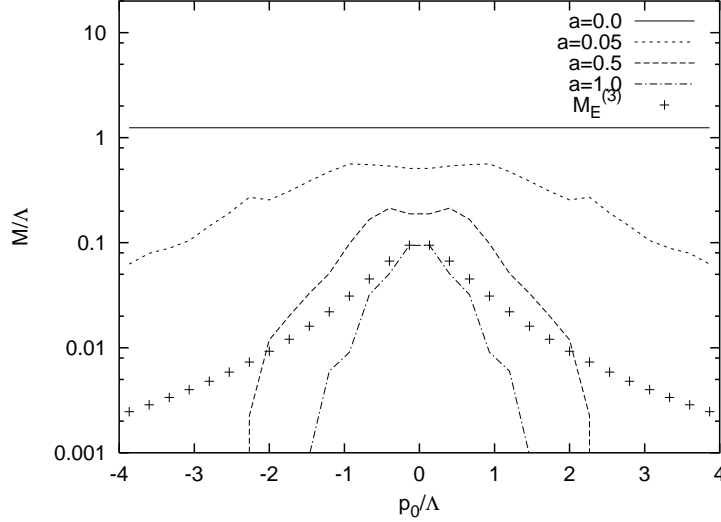


Fig. 2. The p_0 dependence of M/Λ integrated over $0.01\Lambda \leq p \leq \Lambda$ with $a = 0.0, 0.05, 0.5, 1.0$, respectively. The plus symbols represent the calculated results for $M_E^{(3)}$.

For space-like photon exchange with $a = 1$, the mass function $M_1(p_0, p)$ is shown in Fig.4.

§4. Summary and Comments

In this paper, we studied mass functions of fermion solved by the SDE in Minkowski space beyond the IEA. The integration region of the momentum is restricted in space-like photon exchange.

Calculated results are compared with those obtained in Euclidean space. We found these two cases give similar results in which the energy and the momentum are integrated separately, which give larger critical coupling than the case in the IEA.

Our results suggest a correspondence between mass functions of fermion calculated in Euclidean space and those in Minkowski space with space-like photon exchange in QED.

^{*)} Neglecting the mass function in denominator of the fermion propagator, the critical couplings α_c for chiral symmetry breaking given by analytic calculations are $4/(3\pi) < \alpha_c \leq \pi/6$ for Eq.(2.8)⁵⁾ in IEA and $\alpha_c = \pi/3$ for Eq.(3.1),⁷⁾ respectively.

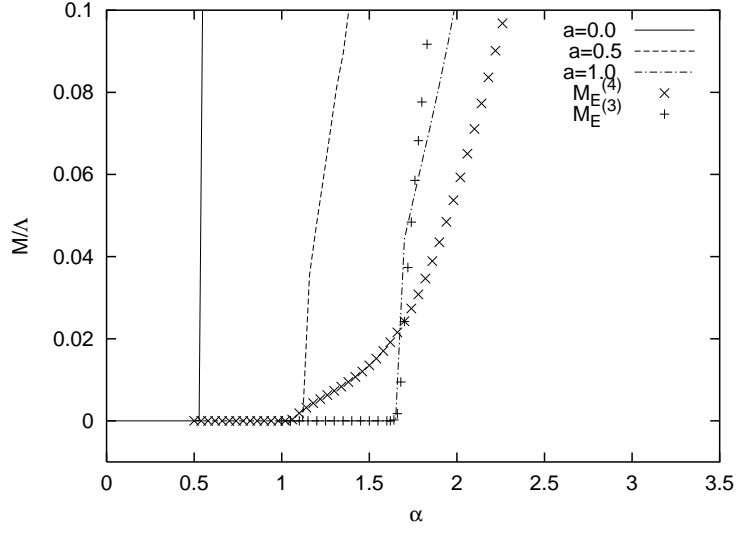


Fig. 3. The α dependence of M/Λ with $a = 0.0, 0.5, 1.0$ integrated over $-4\Lambda \leq p_0 \leq 4\Lambda$ and $0.01\Lambda \leq p \leq \Lambda$. The crossed symbols and the plus symbols denote the results for $M_E^{(4)}/\Lambda$ and $M_E^{(3)}/\Lambda$ in Euclidean space, respectively.

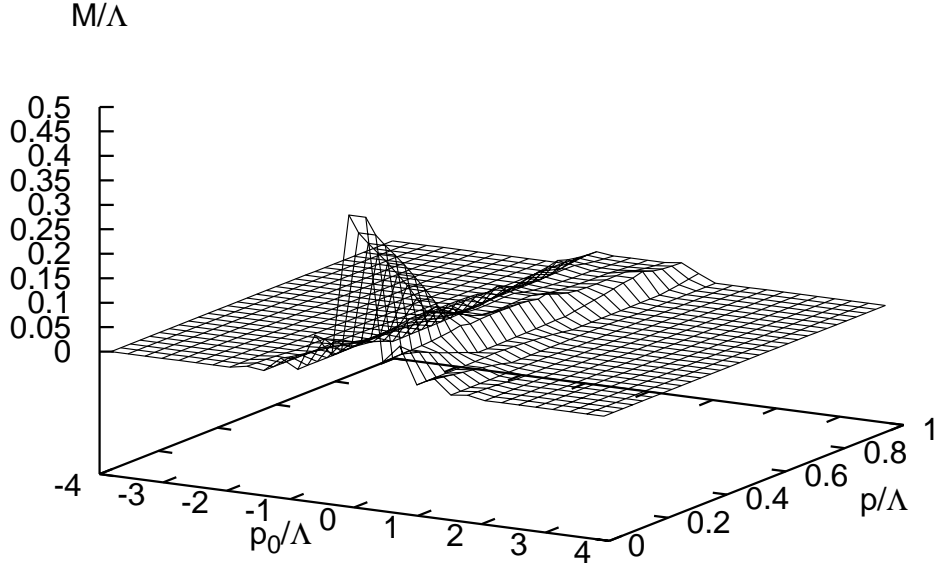


Fig. 4. Three dimensional plot of the mass function $M_1(p_0, p)/\Lambda$ with $\alpha = 2$.

However, in Minkowski space, other contributions with $k^2 \geq 0$ may exist, in which M_2 should be evaluated. In this region, fermion pair productions from virtual photon may contribute. Furthermore, imaginary parts of the mass functions should be evaluated.

Further studies are needed in order to examine mass functions at strong coupling region in Minkowski space.

In future works, we shall include imaginary parts of mass functions and extend our method to QCD and at finite temperature.

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